THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 4: Limits

1. Let A be a subset of $\mathbb{R}$ and let $c \in \mathbb{R}$. Prove that c is a cluster point of A if and only if there exists a sequence $\{a_n\}$ in $A \backslash \{c\}$ such that $\lim_{n\to\infty} a_n = c$.

2. Show that the set $\mathbb{N}$ of natural numbers has no cluster points.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to 0} f(x + c) = L$.

4. Use the $\epsilon - \delta$ definition of limit show that
   (a) $\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = 4$;
   (b) $\lim_{x\to 0} \frac{|x|}{x^2} = 0$.

5. By considering the negation of the $\epsilon - \delta$ definition, show that the following limits do NOT exist.
   (a) $\lim_{x\to 0} \frac{|x|}{x}$;
   (b) $\lim_{x\to 0} \frac{1}{x^2}$.

6. Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\lim_{x\to c}(fg)(x)$ exists, but both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

7. Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ such that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c}(fg)(x)$ exist, but $\lim_{x\to c} g(x)$ does not exist.

8. Show by any method that $\lim_{x\to 0} \sin \frac{1}{x}$ does NOT exist.

9. Let $f : \mathbb{Q} \to \mathbb{R}$ be a function defined by $f(x) = x$. Show that $\lim_{x\to c} f(x) = c$ for any $c \in \mathbb{R}$. (Note: Every real number is a cluster point of $\mathbb{Q}$.)

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{otherwise.} \end{cases}$$

    Show that $\lim_{x\to c} f(x)$ does not exist for any nonzero number c and $\lim_{x\to 0} f(x) = 0$. (Remark: What is the difference between the functions in this question and the previous question?)

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to 0} f(x) = L$, and let $a > 0$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := f(ax)$ for $x \in \mathbb{R}$, show that $\lim_{x\to 0} g(x) = L$.

12. Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x\to c} g(x) = 0$. Prove that $\lim_{x\to c}(fg)(x) = 0$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x \to 0} f(x) = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.

14. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is some $M > 0$ such that

$$|f(x)| \leq M|x|^2 \qquad \text{for all } x \in \mathbb{R}.$$

Prove that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{f(x)}{x} = 0.$$

15. Establish the squeeze theorem at infinity.

16. Let $f : (0, a) \to \mathbb{R}$ for some $a > 0$. Establish a definition of right hand limit of the function f at 0.

17. Let $c \in \mathbb{R}$ and let f be defined for $x \in (c, +\infty)$ and $f(x) > 0$ for all $x \in (c, +\infty)$. Show that $\lim_{x \to c} f(x) = +\infty$ if and only if $\lim_{x \to c} \frac{1}{f(x)} = 0$.